Prerequisite course for actuary exam

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Relation and function

Cartesian plane

What is a Cartesian plane?

What is a relationship?

Plot the following relation on a Cartesian plane.

$$x^2 + y^2 = 16$$

$$X + Y = 4$$

What is a function?

Determine if the following are function:

$y = e^x$

$1 = \frac{x^2}{y}$

$x^2 + y^2 = 4$

$x + y^2 = 4$

Function example

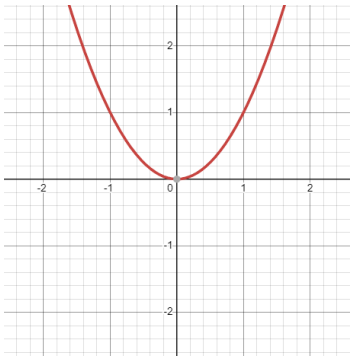
$$X^2 + Y^2 = a^2$$

$$Y = X^2$$

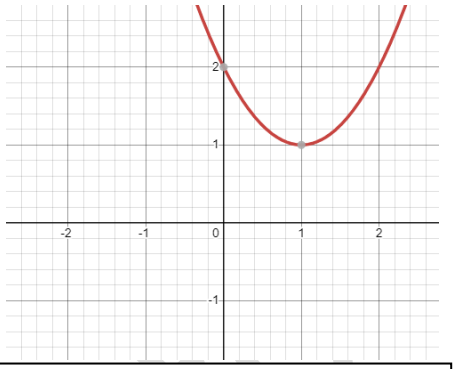
$$y = e^x$$

Function shifting

$$F(x)=y \rightarrow F(x-a)=y-b$$



$F(x) = x^2$



$f(x) - 1 = (X - 1)^2$

Series

Arithmetic series

$$a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d) = na + d \cdot \frac{n(n-1)}{2}$$

Proof:

Sample questions

1. $1 + 2 + 3 + 4 + \cdots + 10 =$

2. $3 + 8 + 13 + 18 + \cdots + 68 =$

3. $2 + 5 + 8 + 11 + 14 + \cdots + 35 =$

Geometric series

For finite terms

$$a + ar + ar^2 + \cdots + ar^{n-1} = a \times \frac{1 - r^n}{1 - r}$$

For infinite terms

$$a + ar + ar^2 + \cdots = \frac{a}{1 - r} \; when \; |r| < 1$$

Proof:

Sample questions

1. $2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \cdots$

2. $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

3. 5+15+45+...+1215

Taylor series

$$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Calculus

Limit

Notation

$$\lim_{x \to a} f(x)$$

Simple example

1. $\lim_{x \to 2} x - 2$

2. $\lim_{x \to 2} \frac{(x-2)(x-3)}{x-2)}$

3. $\lim_{x \to 0} \frac{x^2 - 3x}{x}$

When doesn't limit exist?

- When the right and left sides of a function approach different values.

$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$$

Example:

1. $\lim_{x \to 0} \frac{|x|}{x}$

2. $\lim_{x \to 0} \frac{1}{x^2}$

3. $\lim_{x \to 0} \frac{1}{x}$

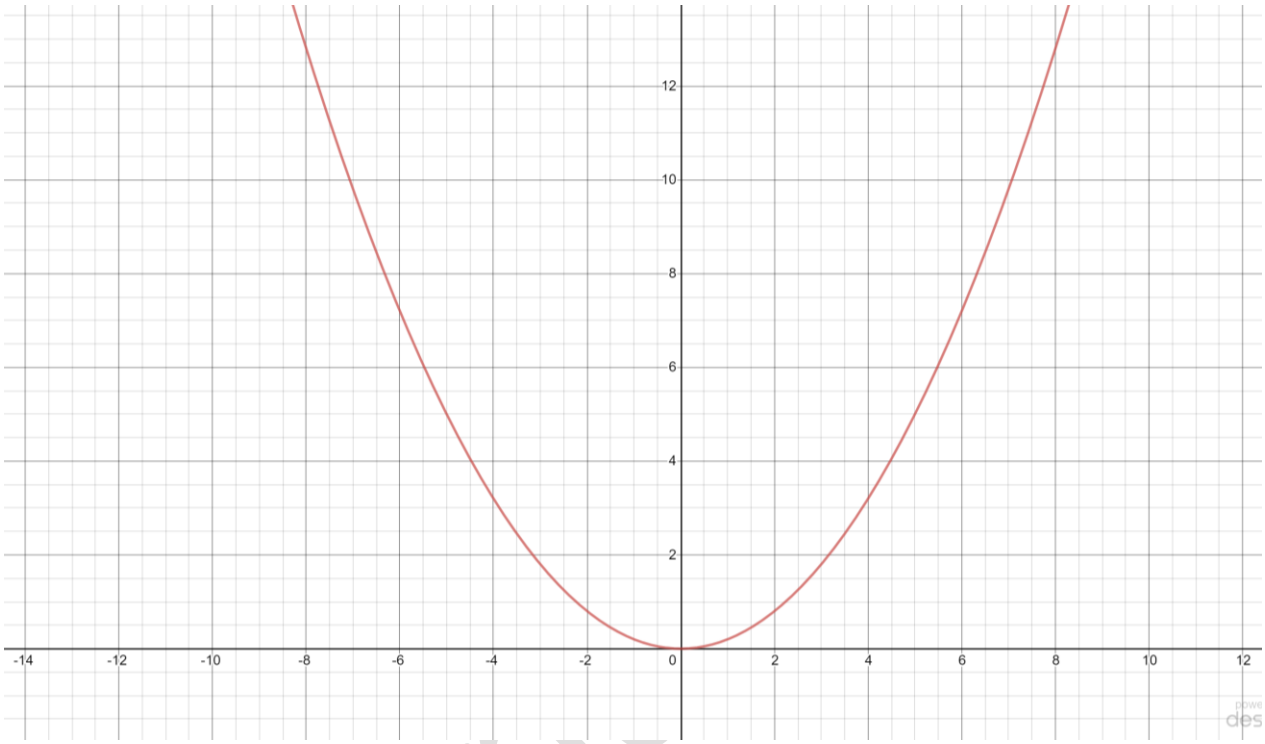
4. $\lim_{x \to 2} \frac{1}{2-x}$

5. $\lim_{x \to 2} \frac{1}{\sqrt{x}}$

Derivative

The derivative of $f(x)$ at $x = x_0$ is denoted $f'(x)$ or $\frac{df}{dx}$

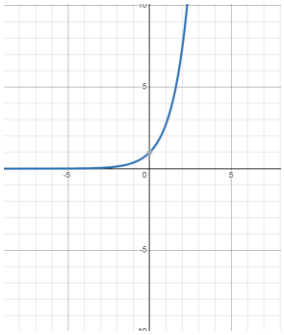
$f'(x)$ is the slope of the graph at point x.



Definition of differentiation:

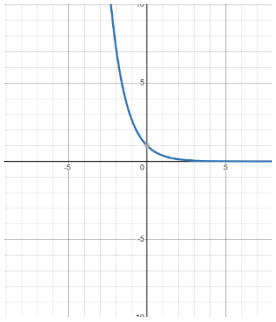
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

first order - $f'(x)$



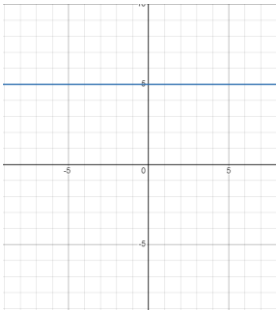
$y = e^x$

$f'(x)>0$



$y = e^{-x}$

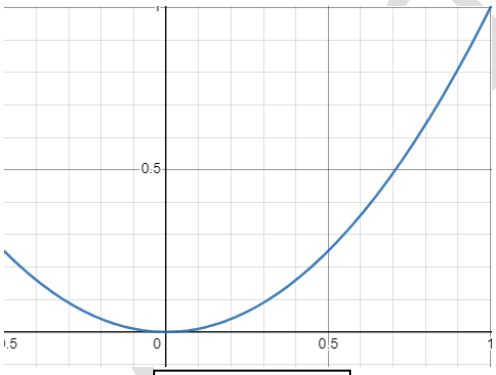
$f'(x)<0$



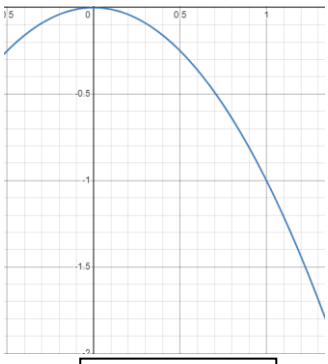
$y = 5$

$f'(0) = 0$

Second order - $f''(x)$



$f''(x)>0$



$f''(x)<0$

continuous and differentiable

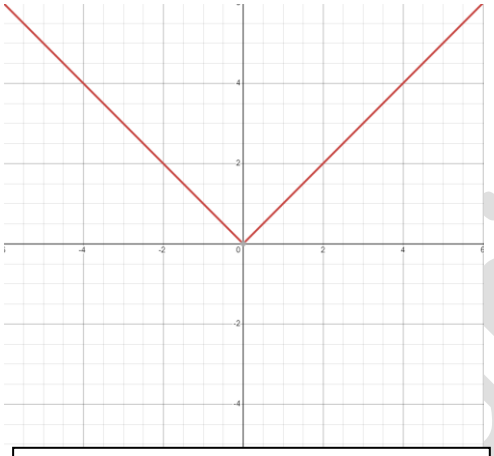
Condition for continuous at point c:

1. $F(c)\ exist$
2. $\lim_{x \to c} f(x) = f(c)$

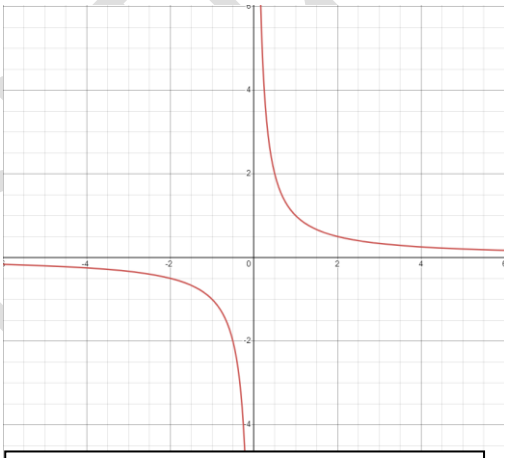
Condition for differentiable at point c:

1. $F'(c)\ exist$

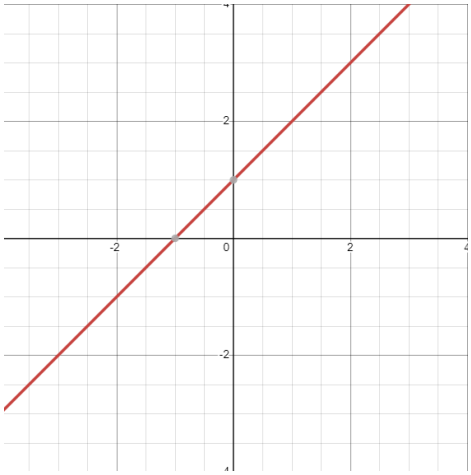
Example



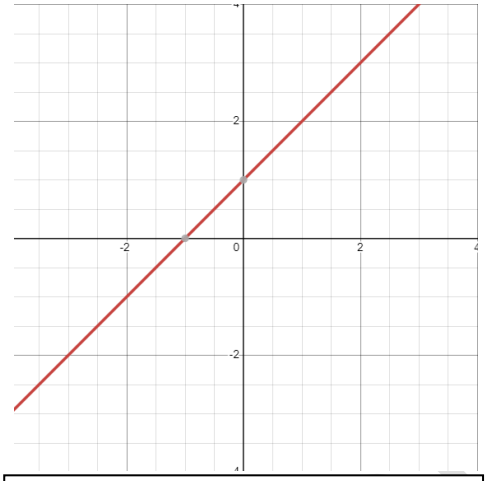
$y = |x|$



$y = \frac{1}{x}$



$$y = \frac{x^2 - 1}{x - 1}$$



$$y = x + 1$$

The Power Rule

$$\frac{d(x^n)}{dx} = nx^{n-1}$$

And

$$\frac{d(C)}{dx} = 0$$

When C is a constant

Sample questions

1. $\frac{d}{dx}x^3 =$

2. $\frac{d}{dx}x^{-4} =$

3. $\frac{d}{dx}8 =$

4. $\frac{d}{dx}x =$

5. $\frac{d}{dx}\sqrt{x} =$

General Derivative rules

$$\frac{\mathrm{d}}{\mathrm{dx}}\big(f(x)+g(x)\big)=f'(x)+g'(x)$$

$$\frac{\mathrm{d}}{\mathrm{dx}}\big(f(x)-g(x)\big)=f'(x)-g'(x)$$

$$\frac{\mathrm{d}}{\mathrm{dx}}\big(cf(x)\big)=c\frac{d}{dx}(f(x))$$

Sample questions

1. $\frac{d}{dx}5x^3=$

2. $\frac{d}{dx}(12x^4+4x^{\frac{3}{2}})=$

3. $\frac{d}{dx}8(x^{-1}+x^{-2})=$

4. $\frac{d}{dx}3\sqrt{x}=$

Product Rule

$$\frac{\mathrm{d}}{\mathrm{dx}}\big(f(x)g(x)\big) = f'(x)g(x) + g'(x)f(x)$$

Sample questions

1. $\frac{d}{dx}(x^3)(x^4) =$

2. $\frac{d}{dx}(x^2+1)(x^4+2) =$

3. $\frac{d}{dx}(x^2+x)(x^2+1) =$

Quotient Rule

$$\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{f(x)}{g(x)}\right)=\frac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}$$

Sample questions

1. $\frac{d}{dx}\frac{x^4}{x^3}$

2. $\frac{d}{dx}\frac{(x^2+1)}{(x^3+2)}$

3. $\frac{d}{dx}\frac{(x^2+x)}{(x^2+1)}=$

Other Derivative

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(a^x) = a^x \log_e a$$

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}$$

Notes

$$In(x) = (\log_e x)$$

Chain Rule

$$\frac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big)\, g'(x)$$

Sample Questions

1. $\frac{d}{dx} e^{2x} =$

2. $\frac{d}{dx} \ln\,(x^3) =$

3. $\frac{d}{dx} (x^2 + \mathrm{x} + 2)^2 =$

4. $\frac{d}{dx} e^{x^2} =$

5. $\frac{d}{dx} \ln\,(e^x + 1) =$

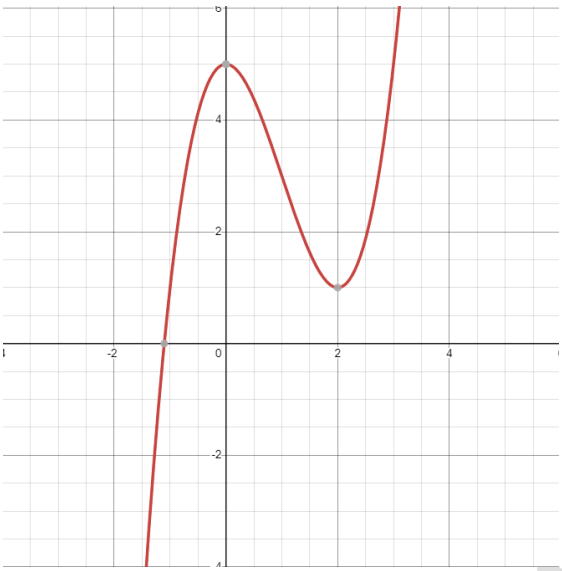
6. $\frac{d}{dx}e^{-5x} =$

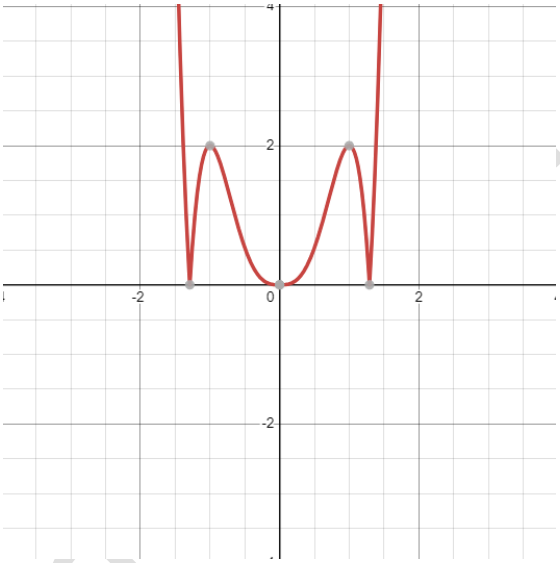
Optimization

Find the maximum and maximum value of a function.

$$f'(x) = 0 \; or \; not \; define, solve \; for \; X$$

Example





$y = x^3 - 3x^2 + 5$	$y = \|x^3\| - 3x^2 + 5$

Indefinite integration

$\int f(x)dx$ mean the antiderivative.

$F'(x) = f(x)$ if and only of $\int f(x)dx = F(x) + c$

Power rule

$$\int \boldsymbol{x^r dx} = \frac{\boldsymbol{x^{r+1}}}{\boldsymbol{r+1}} + \boldsymbol{c}$$

Sample Questions

1. $\int x^9\, dx =$

2. $\int 3x^2 dx =$

3. $\int x^5 dx =$

4. $\int \frac{1}{x^3} dx =$

5. $\int \sqrt{x} dx =$

6. $\int \frac{1}{\sqrt[3]{x}} dx =$

Other integration rule

$$\int (ag(x) \pm bh(x))dx = a\int g(x)dx \pm b\int h(x)\,dx + \boldsymbol{c}$$

$$\int e^x\,dx = e^x + C$$

$$\int \frac{1}{x}dx = In|x| + C$$

Sample Questions

1. $\int \boldsymbol{x^2 + 3x dx} =$

2. $\int \boldsymbol{x^4 + 4 dx} =$

3. $\int \sqrt{\boldsymbol{x}} + \boldsymbol{2x dx} =$

4. $\int \frac{\boldsymbol{x^3+1}}{\boldsymbol{x}}\boldsymbol{dx} =$

5. $\int \boldsymbol{(x+1)(x-1)dx} =$

6. $\int \boldsymbol{3e^x - 2dx} =$

U-Substitution

U-substitute is the inverse of chain rule.

1. Choose a new variable U
2. Compute $\frac{du}{dx}$
3. Replace all x's and dx with u and du
4. Evaluate the integration with variable U
5. Substitute back x in all U

Sample questions

1. $\int (x^3+1)^2 x^2 dx$

2. $\int x^2 e^{\frac{x^3}{4}} dx$

3. $\int x\sqrt{2-x}dx$

4. $\int \frac{x^2}{x^3+5}dx$

Integration by parts

Comes from product rule

1. $\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + f'(x) \cdot g(x)$

2. $f(x) \cdot g'(x) = \frac{d}{dx}[f(x) \cdot g(x)] - f'(x) \cdot g(x)$

3. $\int f(x) \cdot g'(x)dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)dx$

   Or in shorter terms "$\int udv = uv - \int vdu$"

Sample questions

1. $\int xe^{-ax}\,dx$

2. $\int xIn(x)\,dx$

3. $\int In(x)\,dx$

4. $\int \frac{In(x)}{x^2} dx$

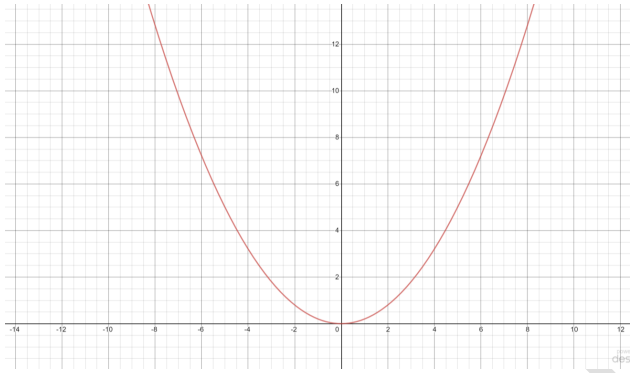
Definite integration

$$\int_a^b f(x)dx$$

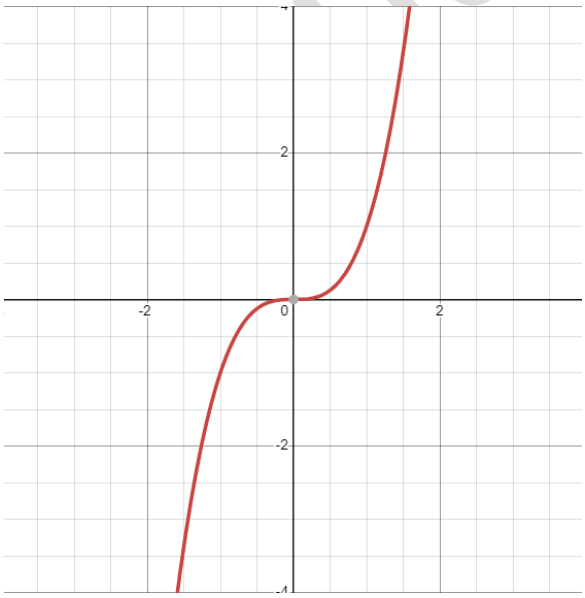
Is the area below the graph from a to b.

Example:

1. $F(X) = x^2$



2. $F(X) = x^3$



Fundamental theorem of calculus, Part 1:
If

$$F(x) = \int f(x)dx$$

Then

$$\int_A^B f(x)dx = F(A) - F(B)$$

Example

1. Find the area below the graph of

$$f(x) = x^3$$

From

$$-1 < x < 3$$

2. Find $\int_{-1}^{3} f(x)dx$

Fundamental theorem of calculus, Part 2:
If

$$F(x) = \int_a^x f(y)dy$$

Then

$$F'(x) = f(x)$$

Example:

1. $f(x) = x^2$

2. $f(x) = x^3$